Full-scale temperature response function (G-function) for heat transfer by borehole ground heat exchangers (GHEs) from sub-hour to decades

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Highlights
- A composite full-scale is built and verified for ground heat exchangers.
- Full-scale model is simplified as multi-stage models to reduce computational cost.
- These models are applicable for time from several minutes to decades.
- Parametric study is performed for ground heat exchangers with single U-tube.

Abstract
Heat transfer by borehole ground heat exchangers involves diverse time-space scales and thus imposes a significant challenge to geothermal engineers. In order to overcome this challenge, this paper develops an analytical full-scale model from the idea of matched asymptotic expansion. The full-scale model is a composite expression consisting of a composite-medium line-source solution (inner solution), a finite line-source solution (outer solution), and an infinite line-source solution. The full-scale model is first verified by a frequency-decomposition method. Furthermore, the full-scale model is reformulated as a multi-stage model based on Duhamel’s theorem to reduce the computational cost. The multi-stage model combines the three separate solutions in a sequential way, i.e., the inner solution for the short-time scale, the conventional infinite line-source solution for the intermediate time scale, and the outer solution for the long-time scale. Finally, we perform a parametric study on a ground heat exchanger with single U-shaped tube, by which the spacing between U-tube legs, the length-to-radius ratio of borehole, the ratios of thermal diffusivities and conductivities of the ground and backfilling material are analyzed.

1. Introduction

Borehole ground heat exchangers (GHEs) are known as critical components in borehole ground heat storage (GHS) and
ground-coupled heat pumps (GCHPs) [1–3]. The purpose of GHS requires that the GHEs must be installed relatively compact for reducing heat loss from the ground storage volume, and the efficiency of GHS will increase with the volume of the GHE cluster; thus GHS prefers large scale applications. In a GCHP system, however, GHEs should be installed as sparsely as possible in order to maximize heat interaction between GHEs and the surrounding ground. Recently, GCHPs have been used increasingly in large buildings. For example, there are GCHPs installed for heating buildings of more than 1,000,000 m² in China; therefore a very large cluster of borehole GHEs is also required for such a GCHP system.

The practical use of the large clusters of GHEs has highlighted the need for an accurate and efficient approach to calculating the thermal process in the ground. From the perspective of accuracy, the heat transfer calculation should use a time resolution ranging from sub-hour to decades, corresponding to a space range from several centimeters to more than one hundred meter (Fig. 1). As a result, an accurate calculation inevitably requires vast computational resources in order to tackle the complete spectrum of the broad time-length scales [4].

The long-term thermal response in the underground process determines the overall feasibility of the GHS and GCHP systems during their application lifetime. The long-time heat build-up in the ground can be as long as the lifecycle of GHEs (~decades) because of the huge heat capacity involved (Fig. 1). Much work has been performed to solve the long-term thermal process underground, including numerical and analytical methods. Numerical methods [5–10] are effective methods of modeling all the underlying thermal processes, but they are computationally too intense for large-scale applications. Thus various analytical models have been proposed and widely used [11–17], among which conventional finite line-source models are the most suitable and efficient for this purpose [11–13].

The thermal response within a borehole can react quickly to variations in thermal loads because of the limited space dimension and heat capacity in the borehole. Scale analysis has shown that the calculation of the temperature within the borehole with a radius \( r_b \sim 5 \text{ cm} \) requires a time resolution in the order of magnitude of an hour [18]. The hourly temperature response is crucial for predicting peak temperatures and thus is vital for the hourly energy analysis and the optimum control of GCHP systems [19]. Predicting the short-term response, however, is more difficult than predicting the long-time process because it is associated with transient heat conduction in a composite medium, together with taking various installations of U-shaped pipes into consideration.

On way to deal with the challenge as mentioned above is to simplify the geometry arrangement in the borehole. A widely used simplification is the equivalent-diameter assumption [20–28], which assumes the U-shaped pipes in a borehole to be a pipe of “equivalent” diameter, thereby reducing the two-dimensional geometry to a one-dimensional hollow cylindrical composite region. The simplified problem can be solved by either generalized orthogonal expansion techniques [20] or the Laplace transform method [22,28]. However, the empirical equivalent-diameter assumption fails to address the thermal influence between legs of the U-shaped pipe and thus leads to an empirical parameter for the design method of borehole GHEs proposed by the ASHRAE [1].

Our group has recently developed an alternative strategy for modeling the short-time response of a GHE based on Jaeger’s instantaneous line-source solution for a cylindrical composite medium [18,29,30]. This approach enables the removal of the equivalent-diameter assumption and can tackle the difficulties associated with both the composite medium and the geometric installations of U-shaped pipes, including single and double U-shaped tubes, W-shaped channels, and spiral-coils. However, the composite-medium line-source model may be one of the most computationally-intensive analytical models because of its complicated mathematical formulation. Moreover, this approach ignores the end effects of GHEs; it is unsuitable for predicting the long-term thermal process.

Thus an efficient analytical model that can address the entire time–space spectrum of the thermal response of borehole GHEs will be beneficial. Claesson and Javed attempted to develop a heat transfer model covering time scales from minutes to decades [25]; but they used the equilibrium-diameter assumption so that a

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**Fig. 1.** Time–space characteristic of borehole ground heat exchangers with U-shaped pipes. The thermal process can span four space scales and eight time scales.
temperature difference exists between the used short- and long-term solutions and an arbitrary temperature shift is proposed at a so-called breaking time. Moreover, the breaking time is empirically determined, without being related to any established theory.

The objective of this paper is to overcome a major obstacle in the calculation of the diverse-scale thermal problem. We build in this paper a full-scale model for the temperature responses of borehole GHEs based on the idea of matched asymptotic expansions. This model can analytically tackle the complexities of time–space characteristic of vertical GHEs with U-shaped tubes. An equivalent form of the full-scale model, called multi-stage model for reducing the computational cost, is also provided. To the best of our knowledge, these two models should be the first two attempts to tackle the effect of composite medium, the time range from sub-hour to decades, as well as the intricate installations of U-shaped pipes in boreholes. More importantly, the two models have a solid theoretical basis; their analytical forms provide an excellent starting point for improving the analysis, design, and simulation of borehole GHEs.

2. Model development

As reviewed above, although various models exist for short-, mid-, or long-term thermal responses of GHEs, a theoretical model suitable for the entire time range is still unavailable. Such a theoretical model is derived here by the idea of matched asymptotic expansions.

2.1. Models for short-, mid-, and long-term response functions

The application of Kelvin’s theory of heat sources to the heat transfer of borehole GHEs leads to an infinite line-source assumption for intermediate time and a finite line-source assumption for long time. The details of the heat-source theory can be found from the classical heat conduction book of Carslaw and Jaeger [31] and Eckert and Drake [32]; thus only final expressions are given below. A characteristic temperature of particular interest is the average temperature on the borehole wall (unit-step load) and has been analytically determined, without being related to any established theory. This form slightly differs from the dimensionless G-function proposed by Claesson and Eskilson [14].

Infinite line-source solution, Eq. (1), suffers from two limitations on the time scale. First, it ignores the influence of the ground and backfilling material in the borehole. The present authors have developed a set of new G-functions for time as short as several minutes based on Jaeger’s composite-medium line-source theory [18,28–30]. A major difference between the composite-medium line-source model and the conventional heat-source models is that the calculated characteristic temperature is not the one on the borehole wall but on the wall of the U-shaped pipe. In the current practice, one or two U-shaped tubes are exclusively used in a borehole, as shown in Fig. 2. The average temperature on the U-pipe wall for these two configurations can be approximated by the average of temperatures at points A and B as labeled in Fig. 2 [18].

By using the temperatures of points A and B, the short-time G function for single U-tube GHEs is defined as [18]

\[
G_t(t) = \frac{1}{2\pi k_0} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \left[ 1 - \exp(-u^2a_0t) \right] \left[ J_{2m}(ur_A) + J_{2m}(ur_B) \right] \left( \frac{u\theta + \psi}{u^2 + \psi^2} \right) du \tag{4a}
\]

and for double U-tube GHEs is [18]

\[
G_t(t) = \frac{1}{2\pi k_0} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \left[ 1 - \exp(-u^2a_0t) \right] \left[ J_{2m}(ur_A) + J_{2m}(ur_B) \right] \left( \frac{u\theta + \psi}{u^2 + \psi^2} \right) du \tag{4b}
\]

where \(a_0\) and \(k_0\) are the thermal diffusivity and conductivity of the backfilling material, respectively; \(r\) is the radius position of the line sources (Fig. 2); \(r_A\) and \(r_B\) are the radius coordinates of points A and B, respectively. \(u\) is the integral variable of dimension of reciprocal of length. In Eqs. (4), the following definitions are used:

\[
G_0(t) = \frac{1}{H} \int_{0}^{H} G_t(t,z) dz. \tag{3}
\]
\[ q(t) = \frac{1}{4\pi \kappa_c} \left\{ \int_0^\infty \frac{\text{erfc} \left( \frac{\sqrt{r_0^2 + z^2}}{2\sqrt{t}} \right)}{H\sqrt{r_0^2 + (z - z_0)^2}} \, dz \, dz' - \text{erfc} \left( \frac{\sqrt{r_0^2 + z_0^2}}{2\sqrt{t_0}} \right) \right\} + \frac{1}{4\pi \kappa_c} \left\{ \int_0^\infty \frac{\text{erfc} \left( \frac{\sqrt{r_0^2 + z^2}}{2\sqrt{t}} \right)}{H\sqrt{r_0^2 + (z + z_0)^2}} \, dz \, dz' - \text{erfc} \left( \frac{\sqrt{r_0^2 + z_0^2}}{2\sqrt{t_0}} \right) \right\}
\]

The composite solution is suitable for the entire time-space spectrum and called full-scale G-function herein. By combining Eqs. (1)–(4), we have a full-scale \( G_c \) function for single U-tube GHEs as follows:

\[ G_c(t) = \frac{1}{4\pi \kappa_c} \left\{ \int_0^\infty \frac{\text{erfc} \left( \frac{\sqrt{r_0^2 + z^2}}{2\sqrt{t}} \right)}{H\sqrt{r_0^2 + (z - z_0)^2}} \, dz \, dz' - \text{erfc} \left( \frac{\sqrt{r_0^2 + z_0^2}}{2\sqrt{t_0}} \right) \right\} + \frac{1}{4\pi \kappa_c} \left\{ \int_0^\infty \frac{\text{erfc} \left( \frac{\sqrt{r_0^2 + z^2}}{2\sqrt{t}} \right)}{H\sqrt{r_0^2 + (z + z_0)^2}} \, dz \, dz' - \text{erfc} \left( \frac{\sqrt{r_0^2 + z_0^2}}{2\sqrt{t_0}} \right) \right\}
\]

and for double U-tube GHE

\[ G_c(t) = \frac{1}{4\pi \kappa_c} \left\{ \int_0^\infty \frac{\text{erfc} \left( \frac{\sqrt{r_0^2 + z^2}}{2\sqrt{t}} \right)}{H\sqrt{r_0^2 + (z - z_0)^2}} \, dz \, dz' - \text{erfc} \left( \frac{\sqrt{r_0^2 + z_0^2}}{2\sqrt{t_0}} \right) \right\} + \frac{1}{4\pi \kappa_c} \left\{ \int_0^\infty \frac{\text{erfc} \left( \frac{\sqrt{r_0^2 + z^2}}{2\sqrt{t}} \right)}{H\sqrt{r_0^2 + (z + z_0)^2}} \, dz \, dz' - \text{erfc} \left( \frac{\sqrt{r_0^2 + z_0^2}}{2\sqrt{t_0}} \right) \right\}
\]

The limits of the double integrals in Eqs. (8) and (9) are ranging from 0 to \( H \).

Once \( G_c \) is determined, the temperature of the circulating fluid due to a unit-step heating/cooling load can be obtained by

\[ T(t) - T_{so} = G_c(t) - \frac{1}{n} R_p \]

where \( T_{so} \) is the initial temperature of the ground; \( R_p \) denotes the thermal resistances of U-shaped pipe:

\[ R_p = \frac{1}{2\pi \kappa_p} \left( \ln \frac{r_o}{r_1} + \frac{k_p}{2\pi r_1} \right) \]

where \( \kappa_p \) is the thermal conductivity of the U-shaped pipe; \( x \) is the convective heat transfer coefficient; and \( r_o \) and \( r_1 \) denote the outer and inner radii of the leg of the U-shaped pipe. For the configuration of single U-shaped pipe \( n \) in Eq. (10) is equal to 2, and for double U-shaped pipe \( n \) is equal to 4. Generally, the thermal resistance due to convective heat transfer is very small compared to that due to heat conduction through the pipe wall and can be ignored [34].

It is nearly impossible to perform an experiment to validate the full-scale \( G_c \) function, which involves time as long as decades. So we theoretically verify this G-function by a method of frequency-decomposition of heat loads. The details of the verification can be found in Appendix A.

3. Multi-stage model

The full-scale solution is theoretically complete, but they are somewhat computationally intensive because the finite line-source model contains a double integral and the composite-medium line-source model involves an infinite series of integrals of the Bessel functions of the first kind and the second kind of order \( n \). This difficulty can be overcome by slightly sacrificing the theoretical completeness, which leads to the following multi-stage model derived from Duhamelet’s theorem.
3.1. Duhamel's theorem

Duhamel’s theorem, which is also referred to as the principle of superposition [31,35], provides a way of solving the problem with time-dependent thermal loads by using the solution of the same problem with time-independent loads instead. Time-varying heating and cooling loads are commonly available in step-wise constant values (Fig. 3). Therefore Duhamel’s theorem, which is originally in the form of a convolution integral, can be reformulated in a discrete form as follows [35]

$$T_i(t) - T_{s0} = \sum_{n=0}^{N-1} \Delta q_n G(\mathbf{x}, t - n\Delta t)$$

where $T_i$ is the temperature at the point under consideration $\mathbf{x}$; $q_n$ equals 0, and $q_n$ is the thermal load during time interval $(n - 1)\Delta t < t < n\Delta t$; $\Delta q_n$ is the step-wise variation of $q$ occurring at times $n\Delta t$; response function $G$ is identical to those defined above [18,35].

We would like to underscore three important conclusions derived from the Duhamel’s theorem. First, this theorem shows that the unit-step response function $G$ is a weight function against the variations of $q$. This conclusion implies that the short- and long-time responses (i.e., $G(\mathbf{x}, t - n\Delta t)$) have an influence on the temperature $T_i(\mathbf{x}, t)$ to the extent of the amplitudes $\Delta q_n$ regardless to the time $t$. Secondly, from the standpoint of modeling, Eq. (12) implies that the G-function needs to only be calculated once for a prescribed $t$ and can be reused within the entire time interval $[0, t]$. Thirdly, Duhamel’s theorem reveals that the G-functions at different time scales affect the final temperature in an independently sequential way, in which G functions involving large $n$ are always multiplied by $\Delta q_n$ associated with the same $n$. This observation suggests that a simple sequential combination of G functions for different time scales should be adequate for predicting the linear problem. However the only question is how to determine the time for switching from one solution to another.

3.2. Multi-stage temperature response functions

In fact, the transition time can be estimated by a scale analysis. According to Bejan [36], the time $t$ can be estimated by the following scaling relation when the transient term in the heat conduction equation is comparable to the diffusive term:

$$t \sim (\Delta r)^2/a$$

Here, $a$ is the thermal diffusivity of the conduction medium; $\Delta r$ is the given space range. The effect of heat capacity can be ignored when $t$ is in the order of magnitude of $10(\Delta r)^2/a$.

For our purpose, two time scales are important. The first one is $t_b = (r_b)^2/a_b$, which is associated with the effect of heat capacity within the borehole; the second time is $t_H = H^2/a_b$, which can be used to estimate the impact of the ground surface. The thermal diffusivities of the ground and backfill material are on the order of magnitude of $10^{-6}$ m²/s. The radius $r_b$ and the length $H$ of a borehole are generally in the order of magnitude of 5 cm and 100 m, respectively. The corresponding time scales $t_b$ and $t_H$ are approximately 1 h and decades, respectively. In general, when $t > 10t_b$, one can switch from the inner solution to the intermediate solution or the outer solution; when $t > 0.01t_H$, one should switch from the intermediate solution to the outer solution. Different from full-scale model, adopting the intermediate or the outer solutions, we need a steady-state borehole thermal resistance $R_b$ [3,37]:

$$T_f(t) - T_{s0} = G_0(t) + R_b$$

or

$$T_f(t) - T_{s0} = G_0(t) + R_b$$

4. Results and discussion

Essentially, the multi-stage model is just a reformulated form of the full-scale model. They should yield identical temperature predictions. The equivalence between them is verified in Fig. 4, which confirms that the difference between these two models is very small and negligible.

Theoretically, the transient solution of composite-medium line source (i.e., the inner solution) should match Eqs. (14) or (15) if the transition time is large enough, since Eq. (16) for $R_b$ is derived from the assumption of steady-state line source. Scale analysis offers only the estimation of the order of magnitude for the transition times; there may be some minor discontinuities when switching from a solution to the other, namely, the discontinuities may occur at the transition times $(10-20)t_b$ and $(0.1-0.01)t_H$. As a result, we suggest the intervals instead of individual values for transition times. For some sets of input parameters, the transition times should be chosen near the upper limits in order to reduce the dis-

Fig. 4. Comparison between full-scale model and multi-stage model.
continuity. In most cases, the discontinuities are very small and have negligible impact on the computation as shown in Fig. 4.

While the multi-stage model and full-scale model are theoretically equivalent, they are different from the perspective of computational algorithm. The composite-medium line-source solution should be only used for resolving short-time or high-frequency temperature variations; it is very computationally intensive as mentioned above. The full-scale model requires the use of this complex composite-medium solution to calculate the high-frequency responses at the entire time domain. In contrast, the multi-stage model only requires the use of this solution within the time $0 < t < (10–20\tau_b)$. Thus the computation time can be reduced.

Some parametric studies have been performed using the full-scale model for borehole GHEs with single U-shaped tube. The temperature variations of the circulating fluid as shown in Figs. 5–7 are subjected to a unit-step thermal load. All the values of the input parameters are labeled in these figures. The data of Fig. 6 are available as part of the online edition (Appendix B) for others to use. The parameters examined herein include the half spacing between legs of U-shaped tube, $D$, the length-to-radius ratio of borehole, $H/r_b$, the ratios of thermal diffusivities and conductivities, $a$ and $k$.

Figs. 5–7 are corresponding to three special values of $D$. These figures show that $D$ does not change the shape of the response curves, but the temperature gradient decreases with increasing $D$. It is because increasing $D$ can enhance the thermal interaction between U-shaped tubes and the surrounding ground. The ratio of thermal diffusivities $a = (a_b/a_s)^{1/2}$ greatly influences the short-time responses. Given the other parameters, the temperature reduces with decreasing $a$, and its influence is significant when the Fo number is smaller than 100, corresponding to a time range of approximately 100 h.

The dimensionless variable $a$ represents the influence of heat capacity of the grouting material, and the half spacing $D$ represents to some extent the quality of the installation of U-shaped tubes. The evaluation of the influence of these factors is intractable problems. As a result of the composite-medium line-source solution, the competitive advantage of the full-scale model is that it has the ability to quantitatively evaluate the impact of $a$ and $D$ on the thermal process. The other analytical models using the equivalent-diameter assumption cannot provide such evaluation [19–25].

On the other hand, the length-to-radius ratio only affects the long-term temperature response. If time is long enough, the temperature rise can reach a quasi-steady state. The temperature of this steady state increases with increasing $H/r_b$, and the time required to approach this state also increases with $H/r_b$ (Figs. 5–7). It should be noted that the $G$ function proposed in the ASHRAE handbook fails to deal with the influence of $H/r_b$ [138]. It is because it uses the infinite cylindrical heat-source model that cannot account for the influence of the ground surface.

The ratio of thermal conductivities of the media, $k = k_s/k_b$, is the most critical parameter. It determines the gradient of the temperature rise, i.e., the rate of the increase of temperature. It also influences the temperature rise during short-, mid-, and long-time. It is unsurprising that the increase of the thermal conductivity of soil (increasing $k$) leads to the reduction of the temperature rise. But an important point to remember is that the curves for different $k$ are yielded with a given constant $a$. We cannot infer from these curves that decreasing $k_b$ (also increasing $k$) can enhance heat
Moreover, Duhamel’s theorem also informs us that for a given simulating time the \( G \) functions only need to be calculated once. Therefore, the first step of a reasonable simulation should be to pre-compute the fluid temperature response to a unit-step heat flux using the full-scale model for a given simulating time (for example, 50 years). Next, the fluid temperature can be reused in terms of Eq. (12) in the subsequent hourly simulation for any time within the simulating time.

We would like to draw attention to the advantages of this new model compared to existing models and its potential impact on the design of borehole GHEs. As emphasized above, the parameters \( a \), \( D \), and \( H/r_b \) have significant influence on the fluid temperature. However, the existing popular design method uses a heat transfer model failing in dealing with these factors accurately \([1,38]\). The impact of the three parameters are either ignored or addressed by some empirical constants. A generally accepted design procedure is to calculate three thermal resistances (i.e., temperature responses) corresponding to three unit-step loads with three time scales, e.g., a ten-year, a one-month, and a several-hour thermal resistances \([1]\). The three thermal resistances and the corresponding heat loads give the temperature response in the ground. As shown in Figs. 5–7, the several-hour thermal resistance depends heavily on the value of \( a \), and the ten-year thermal resistance depends on \( H/r_b \). More accurate evaluation of the influence of these parameters using our new model must lead to more reliable design of GHEs.

Making comparison between the composite-medium solution and a set of sand-box experimental data validates that the solution (i.e., the full-scale model) can predict the temperature response at time as short as several minutes \([29,30]\). The load of a GHE is usually available on an hourly basis; the time resolution of several minutes is sufficient for real applications. Finally the models developed in this paper are only applicable for single GHEs. But they can be extended to address the thermal interaction between adjacent GHEs by the principle of superposition. The accompanying time of this thermal interaction is on the order of several months (Fig. 1). Each GHE in this situation can be simplified as a line of heat source placed in an infinite or finite medium \([11]\). Thus, the
temperature response at each point is the superposition of the responses of all the line sources at the point.

5. Conclusions

Ground heat storage and ground-coupled heat pumps, emerging as sustainable energy technology, have attracted great interests of geothermal engineers and researchers. One of main factors restricting the application of these systems is the difficulty in the calculation of the thermal process in the ground, which can span four space scales and eight time scales (Fig. 1). To clear the way for the GHS and GCHPs applications, this paper develops a full-scale model for heat transfer by borehole ground heat exchangers with U-shaped tubes using a matched asymptotic expansion technique. This model is a composite solution combining the composite-medium line-source solution for short time, the conventional infinite line-source model for intermediate time, and the conventional finite line-source solution for long time. To reduce computational expenses, the full-scale model is rewritten as a multi-stage model. Duhamel’s theorem and scale analysis provide theoretical bases for the multi-stage model. It should be noted that the multi-stage model requires that the conventional heat-source solutions should be used together with a model for borehole thermal resistance ($R_0$). But the full-scale model does not require such feature. Other special features of the full-scale and multi-stage model are listed below:

1. They are analytical models consisting of no empirical simplification.
2. They are applicable for a time range from several minutes to decades, covering all the important time spectrum of the heat transfer by borehole GHEs.
3. They have a strong theoretical foundation, i.e., the idea of match asymptotic expansion, Duhamel’s theorem, and scale analysis. Scale analysis provides estimates for switching times, and Duhamel’s theorem reveals that the simple sequential combination of the models is feasible.

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Appendix A. Verification of full-scale model

In this appendix, we theoretically prove the correctness of the full-scale model by a method of decomposition of heat loads. For this purpose, we adopt the idea from large-eddy simulation (LES) of turbulent flows, which involves multi-scale unsteady fluid motions. In LES, a filtering operation is defined and performed on the governing equation and the initial/boundary conditions [39]; thus the fluid motions are split into the filtered and the residual motions. Such operations can also be performed on the mathematical model of heat transfer by GHEs. It is very direct to prove that the filtering operations will not change the form of the governing equation for heat conduction of GHEs. In contrast to LES, the filtering operations cause no closure problem due to the linearity of the governing equation. The filtering operations change only the form of the time-varying heat-flux boundary condition $q(t)$, by dividing it into filtered low-frequency part and residual high-frequency part [39]:

$$q(t) = \int_{-\infty}^{t} F(t - \tau)q(\tau)d\tau$$

(A1)

where $\tau$ is the filtered heat load, $\tau$ is an integral variable; and $F$ is the filter function. The low-pass Gaussian filter function is used in the following study:

$$F(t) = \sqrt{\frac{6}{\pi A^2}} \exp\left(\frac{6t^2}{A^2}\right)$$

(A2)

where $A$ is filter width. The residual component is $q'(t) = q(t) - q(t)$. Thus, this filtering operation can decompose the original problem into two problems with the same governing equation but subjected to different heat fluxes of different frequencies: the first one refers to the filtered heat load, and the second one refers to the residual heat load. According to the principle of superposition, the solution to the governing problem equals the sum of the solutions to the two decomposed problems.

A trivial but illustrative case is offered herein to demonstrate the equality between this decomposition method and the full-scale model. First, a synthetic time-dependent $q$ containing random signals is generated and then decomposed by using the Gaussian filter function as shown in Fig. 8A. Next, the temperatures due to $\tau$ and $q'$ are obtained respectively by the finite line-source model and the composite-medium model (Fig. 8B), which represent the responses to low- and high-frequency components of $q$. They are then superposed to yield the temperature response to the original $q$ (Fig. 8C). Fig. 8C also shows the temperature yielded by the full-scale method. The two curves shown in Fig. 8C agree very well with each other, demonstrating that the responses determined from the two methods are essentially identical.

The method of frequency decomposition has been proved to be valid when dealing with multi-scale problems (such as LES of turbulent flows); therefore, the equality as shown in Fig. 8C can indirectly verifies the full-scale model.

Appendix B. Supplementary material

The supplementary data is related to Fig. 6 of this article. Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.apenergy.2014.09.013.

References


